Celebrating the Mathematics of Maxim Kontsevich

An event organized by
Friends of the IHÉS and the Simons Foundation

The Ubiquity of Categories

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- Homological Mirror Symmetry and non-commutative geometry

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  - Dynamical systems on categories.

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Categories are a formalization of this insight and have become a universal language in modern mathematics.

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A perfect example of Kontsevich's mathematical vision is his **Homological Mirror Symmetry** (HMS) program.

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Striking Mathematical Consequence: There are different Calabi-Yau geometries X and Y for which

$$\boxed{H^p\left(X, \Omega_X^q\right) = H^p\left(Y, \wedge^q T_Y\right)},$$

and the isomorphism of $H^1(\Omega_X^1)$ and $H^1(T_Y)$ identifies the Yukawa couplings.

Mirror symmetry (ii)

Note: The Yukawa coupling on $H^1(\Omega^1_X)$ depends only on a symplectic structure ω_X while the Yukawa coupling on $H^1(T_Y)$ depends on a complex structure J_Y.

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the Fukaya A_∞ category $\mathbf{Fuk}(X, \omega_X)$ associated with a symplectic structure, and the dg category $\mathbf{D}^b(Y)$ of coherent sheaves associated with a complex structure.

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- The respective categories for a mirror pair are equivalent.
- This equivalence induces isomorphism of cohomology, isomorphism of moduli spaces, and the matching of Yukawa couplings.

The HMS conjecture (i)

Conjecture: (Kontsevich'1994) If $(X, \omega_X, \eta_X) \mid (Y, \omega_Y, \eta_Y)$ be a mirror pair of d-dimensional Calabi-Yau manifolds, then

$$\begin{aligned} \mathbf{Fuk}\,(X, \omega_X, \eta_X) &\cong \mathbf{D}^b(Y) \\ \mathbf{D}^b(X) &\cong \mathbf{Fuk}\,(Y, \omega_Y, \eta_Y) \end{aligned} \qquad (\dagger)$$

and the mirror identifications $H^\bullet(\Omega^\bullet_X) = H^\bullet(\wedge^\bullet T_Y)$ and $H^\bullet(\Omega^\bullet_Y) = H^\bullet(\wedge^\bullet T_X)$ come from the isomorphisms on Hochschild cohomology induced from the equivalences $(\dagger)$.

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Note: Only the grading structure depends on the volume form η_X. Without η_X there is still a version **Fuk**(X, ω_X) of the Fukaya category, which is only $\mathbb{Z}/2$ graded.

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Note: Kontsevich used an equivalent model for $\mathbf{D}^b(X)$ - the category of C^∞ vector bundles on X with integrable $(0, \bullet)$ superconnections. In 2010 Lunts and Orlov proved that all dg-enhancements of $D^b(X)$ are equivalent.

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$$\left(\begin{array}{l}\text{BPS Dirichlet} \\ \text{branes in IIA} \\ \text{theory}\end{array}\right) \overset{\text{HMS}}{\longleftrightarrow} \left(\begin{array}{l}\text{BPS Dirichlet} \\ \text{branes in IIB} \\ \text{theory}\end{array}\right)$$

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In Geometry:

- Conceptualizing the ingredients in HMS lead Kontsvich to new deep understanding of non-commutive geometry.
- The study of moduli spaces of objects in the two categories related by HMS and insights from conformal field theory lead Strominger, Yau, and Zaslow to propose an explicit construction of mirror manifolds:

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Conjecture: (Strominger-Yau-Zaslow'1996) For any mirror pair $(X, \omega_X, \eta_X) \mid (Y, \omega_Y, \eta_Y)$ of Calabi-Yau d-folds there exist $(S^1)^{\times d}$ fibrations of X and Y so that:

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- The deformation away from the large volume/complex structure limits is controlled by (holomorphic or tropical) instanton corrections.

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$\forall\ E, F \in C_X \rightsquigarrow \underline{\mathrm{Hom}}_{C_X}(E,F) \in (\mathrm{Compl}/\mathbb{C})$
so that $\mathrm{Hom}_{C_X}(E, F[i]) = H^i(\underline{\mathrm{Hom}}_{C_X}(E,F))$

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- A symplectic Landau-Ginzburg model $(Y, \mathbf{w}, \omega_Y)$ is also a **nc** space. Here $Y/\mathbb{C}$ is a quasi-projective manifold, $\mathbf{w} : Y \to \mathbb{C}$ is a holomorphic function, ω_Y is a symplectic form on Y, and $C_{(Y,\mathbf{w},\omega_Y)} = \mathbf{FS}(Y, \mathbf{w}, \omega_Y)$ is the Fukaya-Seidel category of Lefschetz thimbles with local systems.

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- (general) Categories of singularities (Orlov): $D^b(Y_0)/\mathrm{Perf}(Y_0)$.

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Theorem: (Bondal -van den Berg'2002) $C_X = \mathbf{D}_{\text{qcoh}}(X)$ has a strong split generator: $E \in C_X$, with

$$\mathbf{D}_{\text{qcoh}}(X) \cong (\text{RHom}(E,E)^{\text{op}} - \text{mod}).$$

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  - $S_X : C_X \to C_X$;
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  A **proper/smooth** **nc** space X is a Calabi-Yau of (fractional) dimension $d = a/b$ if $S_X^b = [a]$.

Proper and smooth nc spaces (i)

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  - scheme, algebraic space, Deligne-Mumford stack;
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(Kontsevich'2001): Fix $L \in \mathrm{Pic}(X)$ - ample, and $\gamma \in \Gamma(\mathrm{tot}(L^\times), \wedge^2 T)^{\mathbb{C}^\times}$ - Poisson structure. Get quantized space $X_\gamma/\mathbb{C}((\hbar))$ with a new homogeneous coordinate ring: $f \star g = fg + \hbar\langle\gamma, df \wedge dg\rangle + \ldots$.

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  - the quantized del Pezzo surfaces of (Artin'1996)

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Calabi-Yau manifolds

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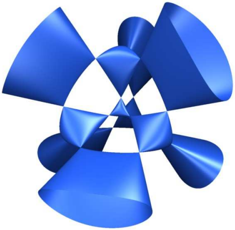
Examples: A hypersurface $X \subset \mathbb{CP}^{d+1}$ of degree $d+2$. For instance we have the quartic Kummer K3 surface



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$$\begin{array}{ccccccc} & & & h^{d,d} & & & \\ & & h^{d,d-1} & & h^{d-1,d} & & \\ & & \cdots & \cdots & \cdots & & \\ h^{d,0} & & h^{d-1,1} & \cdots & h^{1,d-1} & & h^{0,d} \\ & & \cdots & \cdots & \cdots & & \\ & & h^{1,0} & & h^{0,1} & & \\ & & & h^{0,0} & & & \end{array}$$

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- Contraction with $\eta_X \in H^0(\Omega_X^d)$ identifies $\wedge^p T_X$ with Ω_X^{d-p}. If X, Y is a mirror pair of dimension d, then

$$h^{p,q}(X) = h^{d-p,q}(Y).$$

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- $k = h^{1,1}(X)$ is the dimension of the space $\mathcal{M}_{\mathrm{sympl}}(X)$ of complexified Kähler structures ω_X on X.
- $c = h^{2,1}(X) = \dim_{\mathbb{C}} H^1(T_X)$ is the dimension of the space $\mathcal{M}_{\mathrm{cx}}(X)$ of complex structures J_X on X.

Note: $H^1(\Omega^1_X) = T_{\omega_X}\mathcal{M}_{\mathrm{sympl}}(X)$, $H^1(T_Y) = T_{J_Y}\mathcal{M}_{\mathrm{cx}}(Y)$ and the equality

$$H^1(\Omega^1_X) = H^1(T_Y)$$

is the identification of tangent spaces induced by a geometric **Mirror map** isomorphism

$$\mathcal{M}_{\mathrm{sympl}}(X) \cong \mathcal{M}_{\mathrm{cx}}(Y).$$

Back

Yukawa couplings (i)

The mirror isomorphism $H^1(\Omega^1_X) = H^1(T_Y)$ has to identify certain degree d pairings known as **Yukawa couplings**. Assume $d = 3$

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$$\langle \alpha_1, \alpha_2, \alpha_3 \rangle = \int_X \alpha_1 \wedge \alpha_2 \wedge \alpha_3 + \sum_{\substack{\beta \in H_2(X,\mathbb{Z}) \\ \beta \neq 0}} n_\beta \cdot \left(\prod_{i=1}^{3} \int_\beta \alpha_i \right) \cdot \frac{e^{2\pi i \int_\beta \omega}}{1 - e^{2\pi i \int_\beta \omega}}.$$

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n_β: the number (=Gromov-Witten invariant) of holomorphic curves of genus 0 of homology class β

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Mirror symmetry prediction: If $\omega \in \mathcal{M}_{\mathrm{sympl}}(X)$ corresponds to $J \in \mathcal{M}_{\mathrm{cx}}(Y)$ under the mirror map, then under the induced isomorphism $H^1(\Omega^1_X) = H^1(T_Y)$ the pairings $\langle \bullet, \cdots, \bullet \rangle$ agree.

Back